International Journal of Theoretical Physics, Vol. 23, No. 5, 1984

Infinite Conductivity and Perfect Diamagnetism in Type-I Superconductors[1]

G. Papini

Department of Physics and Astronomy, University of Regina, Regina, Saskatchewan S4S 0A2, Canada

Received May 18, 1982

A calculation of the paramagnetic current suggests how infinite conductivity can lead to perfect diamagnetism in soft superconductors in a stationary state.

1. INTRODUCTION

An unsolved question of the theory of soft superconductors regards the relationship between perfect conductivity and perfect diamagnetism. Both properties are required for the phenomenological descriptions of superconductors in a stationary state. They can be shown to be formally independent properties and yet, as stated by Kuper (1978), "the ways they manifest themselves are so closely related that one can hardly believe that they are really independent." Indeed Evans and Rickayzen (1964) showed that in any microscopic theory with an electron scattering mechanism perfect diamagnetism ($\mathbf{B}=0$) is necessary and sufficient for perfect conductivity ($\mathbf{E}=0$). The converse, namely, that $\mathbf{E}=0$ implies $\mathbf{B}=0$, has not so far been proved. If, for instance, one considers the Ginzburg–Landau theory, then $\mathbf{B}=0$ and $\mathbf{E}=0$ are consequences of two long-range order quantities: the phase θ of the wave function and the chemical potential μ related, for a system in equilibrium, by

$$\frac{\partial\theta}{\partial t}=-2\mu/\hbar \tag{1}$$

[1]Work supported in part by the Natural Sciences and Engineering Research Council of Canada.

0020-7748/84/0500-0417$03.50/0

Although equation (1) implies that in steady state conditions μ must be position independent, from which infinite conductivity follows, the converse does not necessarily lead to perfect diamagnetism, which requires $\nabla\theta =$ const.

The purpose of this work is to show that the argument can be turned around and that indeed perfect diamagnetism requires perfect conductivity. This statement is based on a calculation, given in Section 2, of the static paramagnetic response of type-I superconductors proportional to the product of a static scalar potential $G(\mathbf{r})$ and a static magnetic vector potential $A(\mathbf{r})$. Section 3 contains the conclusions.

2. THE STATIC PARAMAGNETIC RESPONSE PROPORTIONAL TO G(r) and A(r).

Let us consider a superconducting slab in the $x-y$ plane and let us choose arbitrarily $\mathbf{a}(-\mathbf{Q})$, the Fourier transform of $\mathbf{A}(\mathbf{r})$, in the y direction. By taking $\mathbf{Q}$ along x the transversality condition is satisfied. In what follows we use the notations of Rickayzen (1959, 1965).

The effects of $\mathbf{A}(\mathbf{r})$ and $G(\mathbf{r})$ on the superconductor can be studied by adding to the free Hamiltonian the perturbations (Rickayzen, 1959, 1965):

$$H_1 = -2\alpha a(-Q)\Sigma k_y\big[l(k,-Q)\big(\gamma^*_{k-Q0}\gamma_{k0} - \gamma^*_{k1}\gamma_{k-Q1}\big) - p(k,-Q)\big(\gamma^*_{k-Q0}\gamma^*_{k1} - \gamma_{k-Q1}\gamma_{k0}\big)\big] + \mathrm{H.C.}$$

and

$$H_2 = G(-Q')\sum_k\big[m(k'-Q')\big(\gamma^*_{k-Q'0}\gamma^*_{k1} + \gamma_{k-Q'1}\gamma_{k0}\big) + n(k'-Q')\big(\gamma^*_{k-Q0}\gamma_{k0} + \gamma^*_{k1}\gamma_{k-Q'1}\big)\big] + \mathrm{H.C.}$$

H_1 represents the usual magnetic interaction term. For simplicity we neglect band structure effects entirely. In addition, Q' is chosen along the z direction and $\alpha = e\hbar/2mc$.

The paramagnetic current operator is

$$\mathbf{J}(\mathbf{Q}'') = \frac{e\hbar}{2m}\sum_k(2\mathbf{k}+\mathbf{Q}'')\big[l(k,Q'')\big(\gamma^*_{k+Q''0}\gamma_{k0} - \gamma^*_{k1}\gamma_{k+Q''1}\big) - p(k'Q'')\big(\gamma^*_{k+Q''0}\gamma^*_{k1} - \gamma_{k+Q''1}\gamma_{k0}\big)\big] + \mathrm{H.C.} \quad (2)$$

with $\mathbf{Q}'' = \mathbf{Q} + \mathbf{Q}'$. The diamagnetic part of the current does not contribute to the total current for the geometry chosen.

In the derivation of the equations of motion we keep only terms which contribute to (2) linearly in the product $a(-Q)G(-Q')$. Since only the Fourier components Q and Q' are assumed to be present for $\mathbf{A}(\mathbf{r})$ and $G(\mathbf{r})$, respectively, we need only the expectation values of products of γ operators which differ by momentum Q to first order in $a(-Q)$ and by momentum Q' to first order in $G(-Q')$. A self-consistent solution for the $\gamma_{k+Q}\gamma_k$ pairs is known and can be found, for instance, in Rickayzen (1959, 1965).

It is represented by

$$\begin{aligned}
\gamma^*_{k+Q0}\gamma^*_{k1} &= -2\alpha a(-Q)k_y(1-f_k-f_{k+Q})p(k,Q)\nu_k^{-1}(Q)\\
\gamma_{k+Q1}\gamma_{k0} &= 2\alpha a(-Q)k_y(1-f_k-f_{k+Q})p(k,Q)\nu_k^{-1}(Q)\\
\gamma^*_{k+Q0}\gamma_{k0} &= 2\alpha a(-Q)k_y(f_k-f_{k+Q})l(k,Q)\tilde{E}_k^{-1}(Q)\\
\gamma^*_{k1}\gamma_{k+Q1} &= -2\alpha a(-Q)k_y(f_k-f_{k+Q})l(k,Q)\tilde{E}_k^{-1}(Q)
\end{aligned} \tag{3}$$

Similarly, a solution for the $\gamma_{k+Q'}\gamma_k$ pairs can be obtained

$$\begin{aligned}
\gamma^*_{k+Q'0}\gamma_{k0} &= 2k_xG(-Q')l(k,Q')\tilde{E}_k^{-1}(Q')(f_k-f_{k+Q'})\\
\gamma^*_{k+Q'0}\gamma^*_{k1} &= -2k_xG(-Q')p(k,Q')\nu_k^{-1}(Q')(1-f_k-f_{k+Q'})\\
\gamma_{k+Q'1}\gamma_{k0} &= 2k_xG(-Q')p(k,Q')\nu_k^{-1}(Q')(1-f_k-f_{k+Q'})\\
\gamma^*_{k1}\gamma_{k+Q'1} &= -2k_xG(-Q')l(k,Q')\tilde{E}^{-1}(Q')(f_k-f_{k+Q'})
\end{aligned} \tag{4}$$

where

$$\nu_k(Q) = E_k + E_{k+Q}$$

and

$$\tilde{E}_k(Q) = E_{k+Q} - E_k$$

by using (3) and (4) one can derive the equations of motion for the γ pairs appearing in (2). The calculations are lengthy but similar to those performed by Miller in connection with the frequency-dependent Hall effect in normal and superconducting metals (Miller, 1961). For the sake of completeness we give the final results

$$\gamma^*_{k+Q''0}\gamma_{k0} = -2\alpha G(-Q')a(-Q)k_y\tilde{E}_k^{-1}(Q'')$$
$$\times\big[m(k+Q',-Q')(1-f_{k+Q'}-f_{k+Q''})p(k+Q',Q)\nu_{k+Q'}^{-1}(Q)$$
$$+m(k+Q'',-Q')(1-f_k-f_{k+Q})p(k,Q)\nu_k^{-1}(Q)$$
$$-n(k+Q',-Q')(f_{k+Q'}-f_{k+Q''})l(k+Q',Q)\tilde{E}_{k+Q'}^{-1}(Q)$$
$$+n(k+Q'',-Q')(f_k-f_{k+Q})l(k,Q)\tilde{E}_k^{-1}(Q)\big]$$

$$\gamma^*_{k+Q''0}\gamma^*_{k1} = -2\alpha G(-Q')a(-Q)k_y\nu_k^{-1}(Q'')$$
$$\times\big[-m(k+Q',-Q')(f_{k+Q'}-f_{k+Q''})l(k+Q',Q)\tilde{E}_{k+Q'}^{-1}(Q)$$
$$+m(k+Q'',-Q')(f_k-f_{k+Q})l(k,Q)\tilde{E}_k^{-1}(Q)$$
$$-n(k+Q'',-Q')(1-f_k-f_{k+Q})p(k,Q)\nu_k^{-1}(Q)$$
$$-n(k+Q',-Q')(1-f_{k+Q'}-f_{k+Q''})$$
$$\times p(k+Q',Q)\nu_{k+Q'}^{-1}(Q)\big] \qquad (5)$$

$$\gamma_{k+Q''1}\gamma_{k0} = 2\alpha G(-Q')a(-Q)k_y\nu_k^{-1}(Q'')$$
$$\times\big[-m(k+Q',-Q')(f_{k+Q'}-f_{k+Q''})l(k+Q',Q)\tilde{E}_{k+Q'}^{-1}(Q)$$
$$+m(k+Q'',-Q')(f_k-f_{k+Q})l(k,Q)\tilde{E}_k^{-1}(Q)$$
$$-n(k+Q',-Q')(1-f_{k+Q'}-f_{k+Q''})p(k+Q',Q)\nu_{k+Q'}^{-1}(Q)$$
$$\times -n(k+Q'',-Q')(1-f_k-f_{k+Q})p(k,Q)\nu_k^{-1}(Q)\big]$$

$$\gamma^*_{k1}\gamma_{k+Q''1} = 2\alpha G(-Q')a(-Q)k_y\tilde{E}_k^{-1}(Q'')$$
$$\times\big[m(k+Q'',-Q')(1-f_k-f_{k+Q})p(k,Q)\nu_k^{-1}(Q)$$
$$+n(k+Q'',-Q')(f_k-f_{k+Q})l(k,Q)\tilde{E}_k^{-1}(Q)$$
$$+m(k+Q',-Q')(1-f_{k+Q'}-f_{k+Q''})p(k+Q',Q)\nu_{k+Q'}^{-1}(Q)$$
$$-n(k+Q',-Q')(f_{k+Q'}-f_{k+Q''})l(k+Q',Q)\tilde{E}_{k+Q'}^{-1}(Q)\big]$$

From (2) and (5) we obtain

$$J_y(Q'') = \frac{e^2\hbar^2}{m^2c}\sum_k k_x k_y^2 G(-Q')a(-Q)L(\varepsilon, \varepsilon_{k+Q'}, \varepsilon_{k+Q''}) \tag{6}$$

with

$$\begin{aligned}
&L(\varepsilon, \varepsilon_{k+Q'}, \varepsilon_{k+Q''}) \\
&= 2l(k,Q'')\Bigg[-\frac{m(k+Q'',-Q')p(k,Q)(1-f_k-f_{k+Q})}{\tilde{E}_k(Q'')\nu_k(Q)} \\
&\quad -\frac{n(k+Q'',-Q')l(k,Q)(f_k-f_{k+Q})}{\tilde{E}_k(Q'')\tilde{E}_k(Q)} \\
&\quad -\frac{m(k+Q',-Q')p(k+Q',Q)(1-f_{k+Q'}-f_{k+Q''})}{\tilde{E}_k(Q'')\nu_{k+Q'}(Q)} \\
&\quad +\frac{n(k+Q',-Q')l(k+Q',Q)(f_{k+Q'}-f_{k+Q''})}{\tilde{E}_k(Q'')\tilde{E}_{k+Q'}(Q)}\Bigg] \\
&\quad -2p(k,Q'')\Bigg[\frac{m(k+Q',-Q')l(k+Q',Q)(f_{k+Q'}-f_{k+Q''})}{\nu_k(Q'')\tilde{E}_{k+Q'}(Q)} \\
&\quad -\frac{m(k+Q'',-Q')l(k,Q)(f_k-f_{k+Q})}{\nu_k(Q'')\tilde{E}_k(Q)} \\
&\quad +\frac{n(k+Q',-Q')p(k+Q',Q)(1-f_{k+Q'}-f_{k+Q''})}{\nu_k(Q'')\nu_{k+Q'}(Q)} \\
&\quad +\frac{n(k+Q'',-Q')p(k,Q)(1-f_k-f_{k+Q})}{\nu_k(Q'')\nu_k(Q)}\Bigg]
\end{aligned} \tag{7}$$

3. B = 0 FROM E = 0

So far no specific form for $G(\mathbf{r})$ has been assumed. It is unnecessary. In fact, since the balancing of the various forms acting on the superelectrons is automatically taken into account by the chemical potential μ^5 (Anderson et al., 1965), we take $G(\mathbf{r}) \propto \mu$. For superconductors, however, stationarity requires that μ be constant far from boundaries. Thus under the assumed conditions,

$$G(-Q') = \mu / k_F \delta(Q') \tag{8}$$

These substitutions transform (6) into the usual expression for the paramagnetic current of a soft superconductor (Kuper, 1964; Rickayzen, 1959, 1965; Tinkham, 1975). This follows from (2) when $Q' = 0$ and can be directly seen by taking the limit $Q' \to 0$ in (6) and (7), as required by (8). Equation (7) yields

$$L(\varepsilon, \varepsilon_1, \varepsilon_2) = \frac{p^2(k,Q)(1 - f_1 - f_2)}{(E_2 + E_1)^2} + \frac{l^2(k,Q)(f_1 - f_2)}{(E_2 - E_1)^2} \tag{9}$$

where 1 and 2 substitute the indices k and $k + Q$, respectively.

By integrating over Q', (6) gives for continuous k

$$J_y(Q) = \frac{e^2 \hbar^2}{m^2 c} \frac{\mu}{k_F} \frac{a(-Q)}{(2\pi)^3} \int d^3k k_x k_y^2 L(\varepsilon, \varepsilon_1, \varepsilon_2) \tag{10}$$

If we now make the approximation

$$\frac{\mu k_x}{E_2 + E_1} \approx \frac{\mu k_x}{E_2 - E_1} = \frac{1}{2} \tag{11}$$

which is equivalent to taking

$$Q \lesssim \frac{2m\Delta}{\hbar^2 k_F} \tag{12}$$

where Δ is the energy gap, equation (10) reduces to the usual paramagnetic current (Kuper, 1968; Rickayzen, 1959, 1965; Tinkham, 1975). Equation (12) obviously preserves superconductivity. The sum of paramagnetic and diamagnetic currents then generates the Meissner effect in a well-known way. In fact, when (11) is verified $L(\varepsilon, \varepsilon_1, \varepsilon_2)$ becomes

$$L(\varepsilon, \varepsilon_1, \varepsilon_2) = \frac{p^2(k,Q)(1 - f_1 - f_2)}{E_2 + E_1} + \frac{l^2(k,Q)(f_1 - f_2)}{E_2 - E_1} \tag{13}$$

If now $\Delta = 0$ as in normal metals, the first term in (13) vanishes while the second term exactly cancels the diamagnetic current. When $\Delta \neq 0$, the first term no longer vanishes, the diamagnetic current is no longer compensated and the total current leads to the Meissner effect. We therefore reach the following conclusion: while θ is responsible for the stiffening of the wave function and, through Δ, for the modulation of (13), μ not only accounts for infinite conductivity, but as shown above, also for the paramagnetic current. It is then the superposition of the two types of order in the total current which results in perfect diamagnetism. In this sense, therefore, perfect diamagnetism requires perfect conductivity.

REFERENCES

Anderson, P. W., Werthamer, N. R., Luttinger, J. M. (1965). *Physical Review*, **138A**, 1157.

Evans, W. A. B., and Rickayzen, G. (1964). *Proceedings of the Physical Society*, **83**, 311.

Kuper, C. G. (1964). *Introduction to the Theory of Superconductivity*. Oxford Univ. Press, London. Sections 1.2 and 4.3 contain a discussion of the problem.

Miller, P. B. (1961). *Physical Review*, **121**, 435.

Rickayzen, G. (1965). *Theory of Superconductivity*. John Wiley and Sons, Inc., New York.

Rickayzen, G. (1959). *Physical Review*, **115**, 795.

Tinkham, M. (1975). *Introduction to Superconductivity*. McGraw-Hill, Inc., New York.